Postbuckling Behavior of a Clamped, Elastically Supported Planar Structure Under Follower Force

Lech Tomski* and Jacek Przybylski†
Czestochowa Institute of Technology, Czestochowa, Poland

The behavior of a geometrically nonlinear planar two-member structure clamped at one end and elastically supported at the other under follower force is studied. The characteristics of forces in the members of the structure during loading vs the rigidity of the supporting spring are discussed. The changes of bending moment at which the system elements interact, the lateral displacement, and tangent slope angle vs loading force are presented. The critical instability (divergence) mechanism of double-bar column is compared with the corresponding single column.

Nomenclature

A_i	= cross sectional area of ith bar ($i = 1,2$)
C_j	= supporting spring rigidity
C_c	= boundary value of supporting spring rigidity
E_i	= modulus of elasticity of ith bar
I_i	= moment of inertia of ith bar
l	= length of the structure
P	= load of the structure
P_c	= maximum value of $P(K_j)$ for boundary value of supporting spring rigidity C_c
$P(K_j)$	= critical load (for supporting spring rigidity C_j)
S_i	= internal axial force of ith bar
$u_i(x)$	= axial displacement of ith bar
$w_i(x)$	= lateral deflection of ith bar
$w_{i,x}(x)$	$\equiv \mathrm{d}w_i(x)/\mathrm{d}x$ = angle between axis x and tangent to the deformed line of ith bar

Introduction

THE instability of a uniform cantilever loaded by a follower or partial follower force has been considered by numerous authors: for example, see Refs. 1–12. In these papers, a column (cantilever) is considered as a continuous system or modeled as a discrete one. Such a column can be of divergence or flutter type, depending on its boundary conditions. Divergence instability of a uniformly clamped, elastically supported cantilever subjected to a follower or partial follower force has been investigated in Refs. 13–16. According to the Euler theory, the straight configuration of the bar is unstable for loads above P_c. A bar loaded with P_c is in neutral equilibrium; no configuration is stable for higher loads.

According to exact elastic theory (EET), however, the load-deflection curve rises indefinitely with a slope that is always positive (or negative for negative deflections), which means that there is a stable configuration for all values of the load. (See Refs. 17–19.)

In solved problems of stability and dynamics, geometrically nonlinear structures are often applied to the moderately large bending theory (MBT).[20,21] (See Ref. 22 for bibliographical information). Special attention must be given to verification of any problem solution based on MBT by use of experimental data. Godley and Chilver[20] have determined the buckling behavior of an initially compressed planar frame, using that theory and its experimental verification. Burgreen[23] and Ray and Bent[24] have presented the solution of nonlinear free flexural vibrations of a beam with immovable ends. In all the above-mentioned works, the moderately large bending theory is in accord with the results of actually performed experiments.

According to the moderately large bending theory as applied in this paper, the equilibrium equations for Hookean material are in the following forms:

In the transverse direction,

$$E_i I_i w_{i,xxxx} - \left[E_i A_i \left(u_{i,x} + \tfrac{1}{2} w_{i,x}^2 \right) w_{i,x} \right]_{,x} = 0 \qquad (1)$$

In the in-plane direction,

$$S_i = -\left[E_i A_i \left(u_{i,x} + \tfrac{1}{2} w_{i,x}^2 \right) \right] \qquad (2)$$

Analysis

This work deals with initial postbuckling behavior of a compound planar clamped, elastically supported structure subjected at the elastically supported end (Fig. 1a) to an axial tangential load.

The structure is constructed from colinear or coplanar prismatic bars of different axial and flexural rigidities. The members are rigidly connected at their ends, both in a displacement and rotational sense.

A physical model of a two-member structure can be presented as a column made of two coaxial tubes (Fig. 1a) or a frame made of a strip located in the center of the structure in which the second member is formed by two identical strips symmetrically located at both sides of the central strip. The structure presented in this paper will be exemplified by the double-tube column (see Fig. 1a), henceforth to be called the double-bar column. The supporting spring stiffness C has its main influence on the behavior of the double-bar column during loading, as well as of an identically supported and loaded single column. The region of divergence instability for the single column is determined in the works of Sundararajan[14] using the dynamic stability criterion and in those of Kounadis[15] using the static stability criterion. From these works, it appears that, at a stiffness lower than the boundary stiffness of spring support C_c, the column loses its stability through flutter and its critical load can be established only by using the dynamic criterion. Therefore, it was observed that the critical

Received April 24, 1985; revision received July 8, 1986.

*Associate Professor, Department of Mechanics and Machine Design Foundations.

†Senior Assistant, Department of Mechanics and Machine Design Foundations.

instability mechanism changed from flutter to divergence, or vice versa, with the changes in the stiffness of the elastic supports. In the case of a double-bar column, the critical instability mechanism differs from that for a single column within the whole range of variability of the supporting spring stiffness C, as will be shown in the following section.

Solution of the Problem

In the case of the coaxial double-bar column under consideration, Eqs. (1) and (2) are accompanied by the following geometrical boundary conditions in the form Fig. 1b:

$$w_1(0) = w_2(0) = 0 \tag{3a,b}$$

$$w_1(l) = w_2(l) = w_l \tag{3c}$$

$$w_{1,x}(0) = w_{2,x}(0) = 0 \tag{3d,e}$$

$$w_{1,x}(l) = w_{2,x}(l) = w_{l,x} \tag{3f}$$

$$u_1(0) = u_2(0) = 0 \tag{3g,h}$$

$$u_1(l) = u_2(l) \tag{3i}$$

and natural conditions:

$$E_1 I_1 w_{1,xx}(l) + E_2 I_2 w_{2,xx}(l) = 0 \tag{4a}$$

$$E_1 I_1 w_{1,xxx}(l) + E_2 I_2 w_{2,xxx}(l) - C w_l = 0 \tag{4b}$$

$$S_1 + S_2 = P \tag{4c}$$

Taking into consideration Eqs. (3a–3c), the solution of Eq. (1) is two equations covering the deflected axes of bars $w_i(x)$,

$$w_i(x) = -\frac{1}{S_i} M_{i0} + \frac{1}{l}\left[w_l + \frac{1}{S_i}(M_{i0} - M_{il})\right] x \tag{5a}$$

$$+ \frac{1}{S_i}\left(M_{il}\frac{1}{\sin k_i l} - M_{i0}\operatorname{ctg} k_i l\right)\sin k_i x$$

$$+ \frac{1}{S_i} M_{i0} \cos k_i x \qquad (i = 1,2) \tag{5b}$$

where:

$$M_{i0} = -E_i I_i w_{1,xx}(0), \qquad M_{il} = -E_i I_i w_{1,xx}(l)$$

$$k_i^2 = S_i / E_i I_i$$

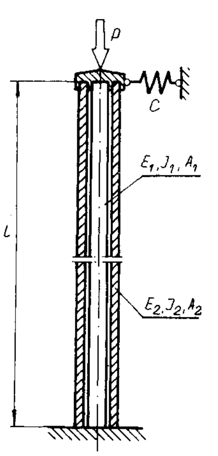

Fig. 1a Double-bar column.

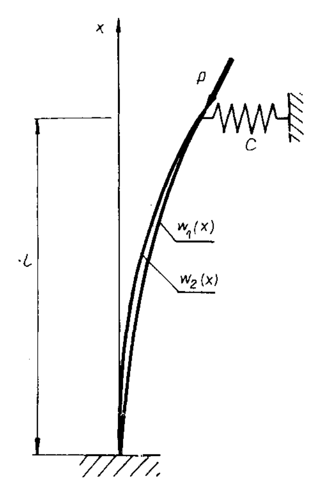

Fig. 1b Deformed axes of both members of the structure under follower force.

Using Eqs. (3d–3f) and (4) for Eqs. (5), we get the system of linear homogenous equations for w_l, M_{10}, M_{1l}, and M_{20},

$$[a_{ij}]\operatorname{col}\{w_l, M_{10}, M_{1l}, M_{20}\} = 0 \qquad i, j = 1,2,3,4 \tag{6a-d}$$

where:

$$a_{11} = a_{34} = a_{42} = 0, \qquad a_{24} = -a_{31} = -a_{41} = -1$$

$$a_{12} = \frac{1}{S_1}\left(1 - \frac{k_1 l}{\sin k_1 l}\right)$$

$$a_{13} = \frac{1}{S_1}(k_1 l \operatorname{ctg} k_1 l - 1) + \frac{1}{S_2}(k_2 l \operatorname{ctg} k_2 l - 1)$$

$$a_{14} = \frac{1}{S_2}\left(\frac{k_2 l}{\sin k_2 l} - 1\right), \qquad a_{21} = Cl,$$

$$a_{22} = \frac{P}{S_1}\left(1 - \frac{k_1 l}{\sin k_1 l}\right) - 1, \qquad a_{23} = \frac{P}{S_1}(k_1 l \operatorname{ctg} k_1 l - 1)$$

$$a_{32} = \frac{1}{S_1}(1 - k_1 l \operatorname{ctg} k_1 l), \qquad a_{33} = \frac{1}{S_1}\left(\frac{k_1 l}{\sin k_1 l} - 1\right)$$

$$a_{43} = \frac{1}{S_2}\left(1 - \frac{k_2 l}{\sin k_2 l}\right), \qquad a_{44} = \frac{1}{S_2}(1 - k_2 l \operatorname{ctg} k_2 l)$$

The nontrivial solution of this set of equations occurs when its determinant is equal to zero. This condition leads to transcendental equation for internal forces S_i in the form,

$$\frac{k_2}{S_2}\left(2\operatorname{tg}\frac{k_2 l}{2} - k_2 l\right)$$

$$\times\left[\frac{k_1 l}{\sin k_1 l}(Cl \cos k_1 l - S_1 - S_2) + S_2 - Cl\right]$$

$$+ \frac{k_1}{S_1}\left(2\operatorname{tg}\frac{k_1 l}{2} - k_1 l\right)$$

$$\times\left[\frac{k_2 l}{\sin k_2 l}(Cl \cos k_2 l - S_1 - S_2) + S_1 - Cl\right]$$

$$+ k_1 k_2 l\left(\frac{\operatorname{tg}(k_1 l/2)}{\operatorname{tg}(k_2 l/2)} + \frac{\operatorname{tg}(k_2 l/2)}{\operatorname{tg}(k_1 l/2)}\right)$$

$$-2\left(k_1 \operatorname{tg}\frac{k_1 l}{2} + k_2 \operatorname{tg}\frac{k_2 l}{2}\right) = 0 \tag{7}$$

Equation (7) makes it possile to find the relationship between the forces S_1 and S_2 and the function of the support stiffness of column C. The sum of forces S_1 and S_2 produces the force P, which loads the system.

Solution of Eqs. (2) with considerations of Eqs. (3g–3i) gives the nonlinear equation for M_{10}, M_{1l}, and M_{20} in the

Table 1 Physical and geometrical data of the column

Parameter	Bar 1	Bar 2
E_i, GPa	100	200
I_i, $m^4 \times 10^6$	2.042	5.27
A_i, $m^2 \times 10^3$	5.0657	3.4557
A_i', $m^2 \times 10^3$	5.0657	23
l, m	4	4

form

$$
\begin{aligned}
&2l\left(\frac{S_2}{E_2A_2}-\frac{S_1}{E_1A_1}\right)\\
&+\frac{1}{S_2^2}\left\{\left[-\frac{1}{l}+\frac{k_2}{2\sin k_2l}\left(\frac{k_2l}{\sin k_2l}+\cos k_2l\right)\right]\right.\\
&\times\left(M_{1l}^2+M_{20}^2\right)\\
&\left.-M_{1l}M_{20}\left[\frac{2}{l}-\frac{k_2}{\sin k_2l}(k_2l\,\mathrm{ctg}\,k_2l+1)\right]\right\}\\
&-\frac{1}{S_1^2}\left\{\left[-\frac{1}{l}+\frac{k_1}{2\sin k_1l}\left(\frac{k_1l}{\sin k_1l}+\cos k_1l\right)\right]\right.\\
&\times\left(M_{10}^2+M_{1l}^2\right)\\
&\left.+M_{10}M_{1l}\left[\frac{2}{l}-\frac{k_1}{\sin k_1l}(k_1l\,\mathrm{ctg}\,k_1l+1)\right]\right\}=0 \qquad (8)
\end{aligned}
$$

Equation (8) together with three linear equations arbitrarily chosen from Eqs. (6a–6d) makes it possible to calculate the values of w_l, M_{10}, M_{1l}, and M_{20} for determined values of forces S_1 and S_2.

Results of Numerical Calculations

Numerical calculations were made for the data listed in Table 1. The results obtained are presented in the form of plots in Figs. 2–6.

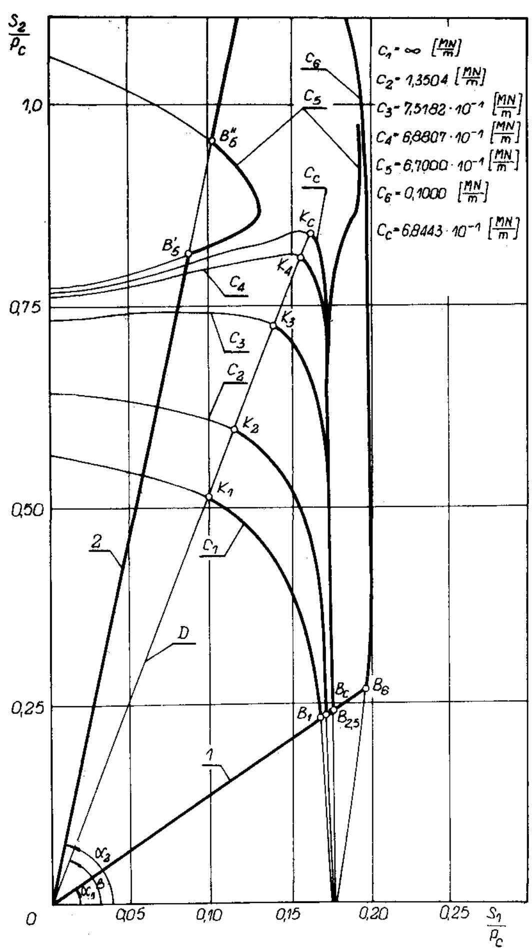

Fig. 2 Relationship between internal forces S_1 and S_2 and support stiffness C_j.

In Fig. 2, the relationship between the internal forces S_i ($i=1,2$) in both bars of the column and the support stiffness of column C_j is presented (the curves are denoted with the symbol of the spring constant C_j).

During loading of the column by force P up to the moment of the start of bending, the linear relationship between forces S_i in both bars is presented by the straight line 1. The slope angle α_1 of this line results from the axial stiffness ratio of both bars,

$$\alpha_1=\mathrm{arctg}\frac{E_2A_2}{E_1A_1} \qquad (9)$$

A further increase of the external force P causes the increase of internal forces S_i from the point B_j (points $B_{1\text{-}4}$ and B_c), where the bending moment between the bars of column begins

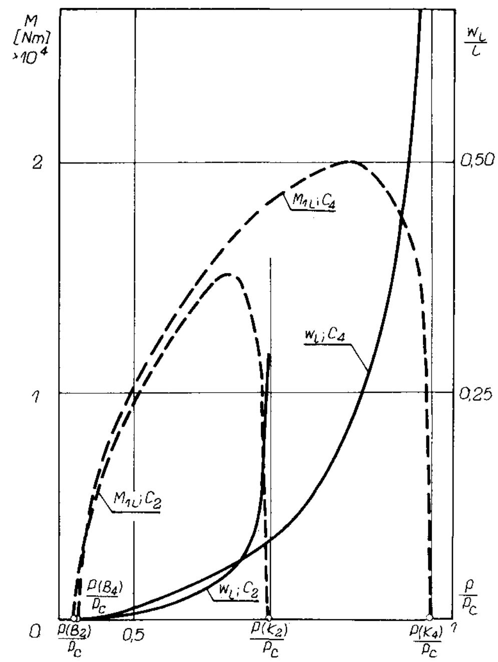

Fig. 3 Bending moment M_{1l} and transverse deflections w_l for two values of constants C_2 and C_4 of spring support force P.

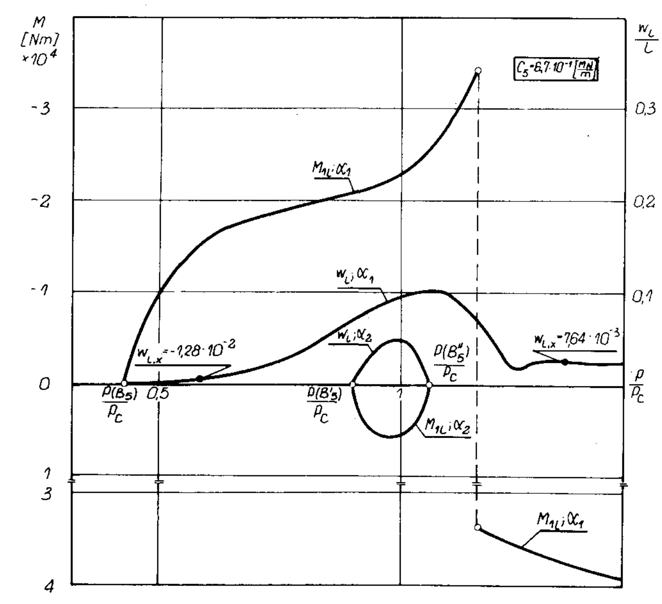

Fig. 4 Bending moment M_{1l} and transverse deflection w_l for spring constant C_5 vs force P.

to appear (Fig. 3), to the point K_j (points $K_{1\text{-}4}$ and K_c) that corresponds to the critical load for a single column of stiffness $EI = E_1I_1 + E_2I_2$. Points K_j and K_c are located on the common straight line D inclined to the horizontal line at the angle β (Fig. 2).

$$\beta = \operatorname{arctg} \frac{E_2 I_2}{E_1 I_1} \tag{10}$$

According to Eq. (10) the following relation must exist at point K_j (Fig. 2):

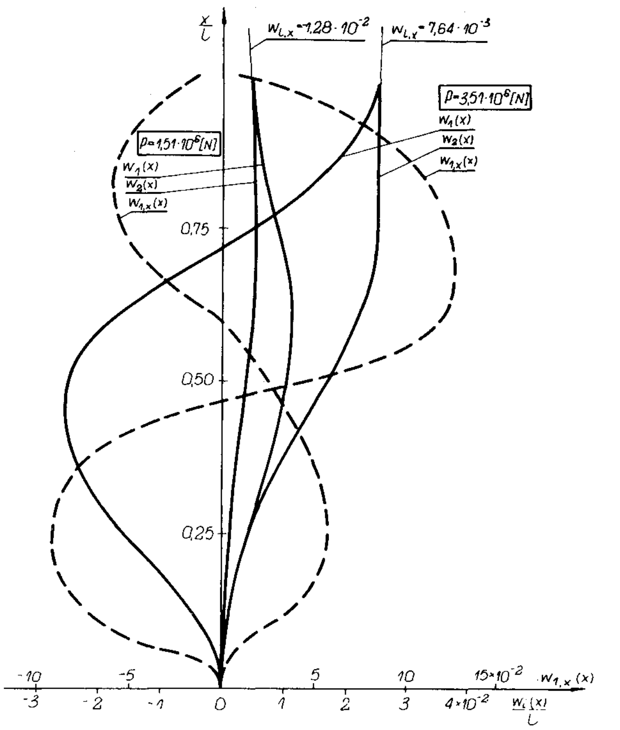

Fig. 5 Transverse displacement $w_i(x)$ (solid line) of the bars and tangent slope angle $w_{1,x}(x)$ (broken line) along bar 1.

$$\frac{S_2(K_j)}{S_1(K_j)} = \frac{E_2 I_2}{E_1 I_1} \tag{11}$$

where $S_i(K_j)$ is the value of the force S_i $(i = 1,2)$ at point K_j. Considering Eqs. (11) and (4c), we have

$$\frac{S_1(K_j)}{E_1 I_1} = \frac{S_2(K_j)}{E_2 I_2} = \frac{P(K_j)}{E_1 I_1 + E_2 I_2} = k^2 \tag{12}$$

where $P(K_j)$ is the critical value of the force P at point K_j. Inserting Eq. (12) into Eq. (7), we get the buckling equation for a single column in the form,[15]

$$\frac{kl}{\sin kl}\left[Cl \cos kl - P(K_j)\right] - Cl = 0 \tag{13}$$

The run of internal forces S_i described above occurs when the stiffness of the support of column end C_j is within the range of $C_c \le C_j \le C_1$ (curves C_1-C_4 and C_c in Fig. 2), where C_1 is the stiffness equal to infinity (the case of a fixed, supported column).

For a single elastically supported cantilever subjected to a follower force, divergence instability occurs at $C \in \langle C_c, \infty)$. For $C \in \langle 0, C_c)$, the only stable configuration for the cantilever is the straight-line configuration and the column loses its stability through flutter.

Using the moderately large bending theory for a double-bar column, when $C_j \in \langle C_c, \infty)$, we get the critical load $P(K_j) = S_1(K_j) + S_2(K_j)$. The internal forces S_i change with the increase of the external loading force P within the range $\langle 0, S_i(K_j)\rangle$. When the support stiffness equals C_c, we get a maximum value for critical load P_c (point K_c in Fig. 2), where $P_c \equiv P(K_c) = S_1(K_c) + S_2(K_c)$.

For the stiffness $0 \le C_j < C_c$, there is a different characteristic for internal forces S_i during the loading of the column with a force P. There is no K_j point on curves C_j (see curves C_5 and C_6 in Fig. 2). With the increase in external force P, a relationship between forces S_i exists initially along straight line 1 ($\alpha_1 < \beta$) to points B_5 or B_5 (the column is compressed only) and then along curve C_5 or C_6 (where the column is bent and compressed).

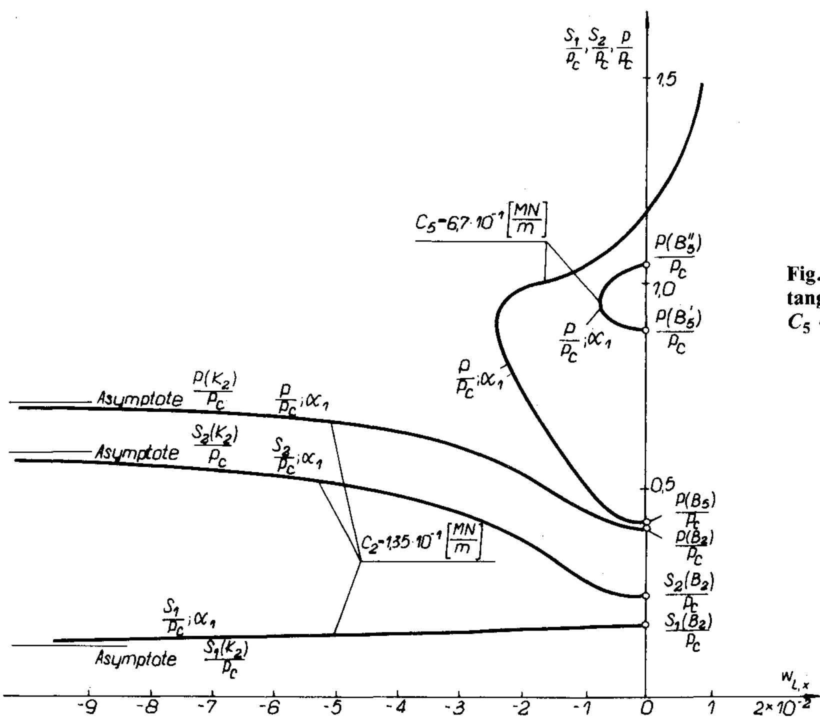

Fig. 6 Characteristics of force P and forces S_1 and S_2 vs tangent slope angle $w_{l,x}$ for two values of constants C_2 and C_5 of spring support.

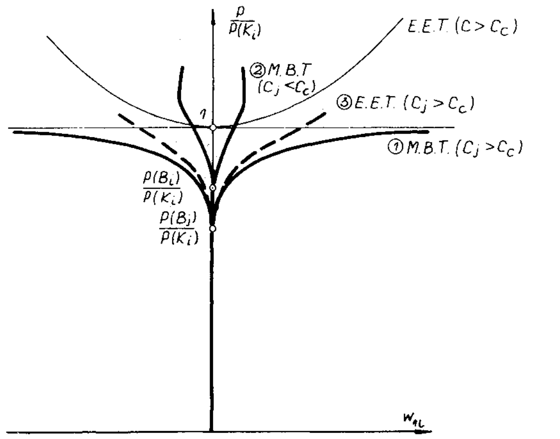

Fig. 7 Comparison of equilibrium paths of cantilever and double-bar column.

When the ratio of axial stiffness is changed,

$$\alpha_2 = \text{arctg}\frac{E_2 A_2}{E_1 A_1'} > \beta \tag{14}$$

and the values of the flexural rigidity of particular bars are substituted without changes ($E_1 I_1$, $E_2 I_2$), the characteristics of internal forces S_i in both bars of the column will be initially along straight line 2 (sloped at the angle α_2) to the point B_5 and then the column will be bent (relationship between forces S_i along curves B_5' and B_5''). Further increases in external loading will cause the next pass from the curve of bending on straight line 2. Equation (7) has no solution for decreasing force P.

Figure 3 presents a plot of the displacement of the elastically supported end of column (w_l solid line) and of the bending moment of the interacting bars (M_{1l} broken line) for two values of the spring supporting columns C_2 and C_4 ($C_c < C_2 < C_1$, $C_c < C_4 < C_1$) vs force P loading the column. The bending moment appears when the external force exceeds the value $P(B_2)$, $[P(B_4)]$ and then takes on the maximum value and is equal to zero for the value $P(K_2)$ $[P(K_4)]$, where the external force reaches the value of the critical load. The displacement w_l also appears above point B_2 (B_4) and then increases to infinity at point K_2,(K_4).

The changes of moment M_{1l} and displacement w_l vs force P for the supporting spring constant C_5 ($C_5 < C_c$), taking into consideration the different ratios of the axial rigidities of both bars (α_1 and α_2), are presented in Fig. 4. For the ratio α_1, displacement w_l and moment M_{1l} increase from zero (point B_5 in Fig. 2). After reaching the maximum, displacement w_l decreases, but moment M_{1l} changes its sign (step on the plot), although its absolute value increases with the increase in external force P. Two marked points on the curve (w_l and α_1) correspond to two forces $P = 1.515 \times 10^6$ and 3.513×10^6 N acting in directions $w_{1,x} = -1.28 \times 10^{-2}$ and 7.64×10^{-3}, respectively. For the ratio α_2, both moment M_{1l} and displacement w_l exist only between points B' and B'' (Fig. 2).

Plots of transverse displacements $w_i(x)$ of both bars (solid line) and exemplary plots of the angle between axis x and tangent to the deformed line of the centroid of bar 1 [$w_{1,x}(x)$, broken line] are presented in Fig. 5 for two values of the force P mentioned above. Because of the flexural rigidity $E_2 I_2 > E_1 I_1$, higher transverse displacements occur in bar 1. For the force $P = 3.513 \times 10^6$ N, the transverse displacements $w_1(x)$ change their sign along the length of the bar.

The characteristics of external force P vs slope $w_{l,x}$ for two rigidities of support C_2 and C_5 ($C_c < C_2 < C_1$, $C_5 < C_c$) are plotted in Fig. 6. In addition, the changes of forces in bars S_1 and S_2 are presented for rigidity C_2. In this case, as force P increases to the value of the critical load, only the value of the force in bar 2 increases, while the force in bar 1 decreases ($E_2 I_2 > E_1 I_1$). For rigidity C_5, the characteristics of external force P depend on the ratio of axial rigidity.

Conclusions

On the basis of the papers cited and the results obtained in this work, the equilibrium path for elastically supported single-cantilever (thin curves) and double-bar columns (planar structure, thick curves) subjected to a follower force at the elastically supported end are schematically presented in Fig. 7. In the case of the double-bar column, because of the interaction of both members, the bifurcation point appears at load $P(B_j)$, which is less than the critical (bifurcating) load $P(K_j)$ for the corresponding single column. For force P leading to $P(K_j)$ (when $C_j > C_c$, supporting spring stiffness greater than the bound stiffness), transverse displacement of the column leads to infinity (curve 1).

For $C_j < C_c$, a gradual increase of the displacement occurs at the same time as the growth of force P (curve 2 in Fig. 7). Applying the moderately large bending theory (MBT), we can describe the initial postbuckling behavior of the double-bar column.

We may foresee that the application of the exact elastic theory (EET) will lead to an explanation of the postbuckling behavior of such a column for $P > P(K_j)$ (curve 3).

References

[1]Bolotin, V.V., *Nonconservative Problems of the Theory of Elastic Stability*, Nauk, Moscow, 1961 (English translation published by Pergamon, New York, 1963).

[2]Bolotin, V.V., and Zhinzher, N.T., "Effects of Damping on Stability of Elastic Systems Subjected to Nonconserative Forces," *International Journal of Solids and Structures*, Vol. 5, 1969, pp. 965–989.

[3]Herrmann, G., "Stability of Equilibrium of Elastic Systems Subjected to Nonconservative Forces," *Applied Mechanics Reviews*, Vol. 20, 1967, pp. 103–108.

[4]Herrmann, G. and Bungay, R.W., "On the Stability of Elastic Systems Subjected to Nonconservative Forces," *Jounal of Applied Mechanics*, Vol. 31, Sept. 1964, pp. 435–440.

[5]Kordas, Z. and Zyczkowski, M., "On the Loss of Stability of a Rod under a Super-Tangential Force," *Archiwum Mechaniki Stosowanej*, Vol. 1, No. 15, 1963, pp. 7–31.

[6]Leipholz, H., "Anwendung des Galerkinschen Verfahrens auf nichtkonservative Stabilitatsprobleme des elastischen Stabas," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 13, 1962, pp. 359–372.

[7]Nemat-Nasser, S., "Thermoelastic Stability under General Loads," *Applied Mechanics Reviews*, Vol. 23, 1970, pp. 615–624.

[8]Pedersen, P., "Influence of Boundary Conditions on the Stability of a Column under Non-Conservative Load," *International Journal of Solids and Structures*, Vol. 13, 1977, pp. 445–455.

[9]Qiu, Z.L. and Nemat-Nasser, S., "Instability of an Articulated Cantilever Induced by an Impinging Airjet," *International Journal of Solids and Structures*, Vol. 21, 1985, pp. 145–154.

[10]Ziegler, H., "Linear Elastic Stability," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 4, 1953, pp. 89–184.

[11]Ziegler, H., "On the Concept of Elastic Stability," *Advances in Applied Mechanics*, edited by H.L. Dryden and T. von Kármán, Vol. 4, 1956, pp. 351–403.

[12]Ziegler, H., *Principles of Structural Stability*, Blaisdell, Waltham, MA, 1968.

[13]Kordas, Z., "The Stability of an Elastically Clampled Compressed Bar in the General Case of Behavior of Load," *Rozprawy Inzynierskie*, Vol. 3, No. 11, 1963, pp. 435–448.

[14]Sundararajan, C., "Influence of an Elastic End Support on the Vibration and Stability of Beck's Column," *International Journal of Mechanical Sciences*, Vol. 19, 1976, pp. 239–241.

[15]Kounadis, A.N., "The Existence of Regions of Divergence Instability for Nonconservative Systems under Follower Forces," *International Journal of Solids and Structures*, Vol. 19, No. 8, 1983, pp. 825–833.

[16]Tomski, L. and Przybylski, J., "Static Instability of Elastically Restrained Cantilever under Partial Follower Force," *AIAA Journal*, Vol. 23, Oct. 1985, pp. 1637–1639.

[17]Thompson, J.M.T. and Hunt, G.W., *A General Theory of Elastic Stability*, Wiley, London, 1973.

[18]Budiansky, B., "Theory of Buckling and Post-Buckling Behavior of Elastic Structures," *Advances in Applied Mechanics*, Vol. 14, 1974, pp. 1–65.

[19]Ashwell, D.G., "Nonlinear Problems," *Handbook of Engineering Mechanics*, edited by W. Flügge, McGraw-Hill, New York, 1962, pp. 45-1–45-19.

[20]Godley, M.H.R., and Chilver, A.H., "Elastic Buckling of Overbraced Frames," *Journal of Mechanical Engineering Science*, Vol. 12, No. 4, 1970, pp. 238–247.

[21]Kounadis, A.N., Giri, J., and Simitses, G.J., "Nonlinear Stability Analysis of an Eccentrically Loaded Two-Bar Frame," *Journal of Applied Mechanics*, Vol. 44, Dec. 1977, pp. 701–706.

[22]Bhashyam, G.R. and Prathap, G., "Galerkin Finite Element Method for Non-Linear Beam Vibrations," *Journal of Sound and Vibrations*, Vol. 72, No. 2, 1980, pp. 191–203.

[23]Burgreen, D., "Free Vibrations of a Pin-Ended Column with Constant Distance between Pin Ends," *Journal of Applied Mechanics*, Vol. 18, June 1951, pp. 135–139.

[24]Ray, J.D. and Bert, C.W., "Nonlinear Vibrations of a Beam with Pinned Ends," *Journal of Engineering for Industry*, Vol. 91, Nov, 1969, pp. 997–1004.